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Semidispersive wave systems

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The statistical initial-value problem for a class of weakly coupled waves whose linear dispersion relation is $\omega \propto \pm |\mathbf{k}|$ is examined. It is found that in two and higher dimensions a natural asymptotic closure is possible. The redistribution of energy is achieved by means of two mechanisms; the first by a resonance between collinear wave vectors; the second by a local transfer between adjacent rays. The entropy functional is $\int \log n(\mathbf{k})\, d\mathbf{k}$ and corresponds to particles obeying Bose–Einstein statistics.

1. Introduction and general discussion

The weak interaction problem in continuum physics has received much interest in recent years and particular attention has been paid to weakly non-linear dispersive wave systems. The result of such investigations has usually been the derivation of a Boltzmann-like equation which, in the context of continuum mechanics, describes the redistribution of spectral energy (analogous to number density) amongst the various scales. Some authors (Elsasser & Graff, private communication) use the generating functional for the Fourier moments and describe its temporal evolution. In earlier work (for example, Litvak 1960, Hasselmann 1962, Galeev & Karpman 1963), the usual procedure was to invoke some *ad hoc* statistical hypothesis in order to close the hierarchy of moment equations. It was later shown by Benney & Saffman (1966) that this hypothesis was, in fact, unnecessary. The decay of the zeroth-order (in the perturbation scheme) third and higher cumulants, due to the natural incoherence of dispersive waves, is sufficient to induce a natural asymptotic closure on the system. The reason for the closure is that the cumulant (or moment) evolution separates into two processes. The first process occurs on a time scale given by a characteristic wave period and involves a decoupling of initial correlations due to the dispersive nature of the waves and an approach to the Gaussian state as one might expect from the central limit theorem. The second process, occurring over longer time scales given by a characteristic wave period divided by powers of the measure of weak coupling, is one of regeneration of these higher cumulants by products of lower-order ones. These terms supply the energy transfer mechanisms which occur and it is their structure which enables us to close the system. Certain controversial questions raised by Hasselmann (1967) involving the consistency of the Benney–Saffman

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analysis have been answered by recent work of Newell (1968) and Benney & Newell (1969).

Thus, achieving a natural closure is the result of the processes of forgetting and remembering being disjoint in the sense that they operate on different time scales. A necessary ingredient for this success is a sufficient degree of decoupling of initial correlations by the linear response of the system. It has long been thought that non-dispersive waves do not have this property due to the fact that, since all disturbances travel with the same speed, initial correlations are kept intact. This is true to some degree, but a finer definition of what is and what is not dispersive is required. Since the energy and momentum travel with the group velocity, one expects initial correlations to be preserved in systems where the group velocity is a constant vector. Examples of such waves are Alfvén waves and one-dimensional, shallow-water and sound waves. But in two and higher dimensions, the familiar dispersion relation

$$\omega \propto \pm |\mathbf{k}| \tag{1.1}$$

(where $\mathbf{k}$ is the wave vector and ω the frequency) is in some sense dispersive, as the group velocity although constant in magnitude depends on the wave direction. Thus, while it is true that along a given ray in $\mathbf{k}$ space correlations are preserved, the fact that many wave groups, carrying statistically independent information, cross a given ray leads to an approach to the Gaussian state. The result is that the zeroth-order (in the perturbation scheme) higher cumulants decay with time. The central limit theorem is again operative and a natural closure can result.

The direct similarity with dispersive systems ends at this point. The mechanism by which dispersive waves redistribute their energy is one of resonance; given a wave vector $\mathbf{k}$, there is a non-trivial locus of wave vectors $\mathbf{k}_1$, such that $\pm\omega(\mathbf{k}) \pm \omega(\mathbf{k}_1) = \pm\omega(\mathbf{k}+\mathbf{k}_1)$. It turns out that in the present case the functional

$$h = \pm\omega(\mathbf{k}) \pm \omega(\mathbf{k}_1) \pm \omega(\mathbf{k}+\mathbf{k}_1) \tag{1.2}$$

has a double zero. This affects significantly the asymptotic behaviour of certain singular integrals of the form

$$\int_{-\infty}^{\infty} F(\mathbf{k}_1, \mathbf{k})\, \Delta(h)\, d\mathbf{k}_1, \quad \Delta(h) = \frac{e^{iht}-1}{ih}, \tag{1.3}$$

whose evaluation is required in obtaining the closure equations. It turns out that the asymptotic behaviour of the real part of the Δ function acts like a Dirac delta function which in the three-dimensional case can be evaluated as $\delta(h)$. Thus, in a three-dimensional conservative system (where only the real part of Δ appears in the energy transport equation), the result derived in § 2 can be formally derived from the fully dispersive case. However, the similarity is only formal and justification for its use can only be given *a posteriori* as the ordering scheme and closure equations depend upon the total asymptotic behaviour of the Δ function and the resulting long-time behaviour of all the cumulants. Both are significantly different in the present situation. In two dimensions, the structure of the long-time be-

haviour of the Δ function is very different and cannot be related at all to the dispersive case. The re-ordering of the perturbation series is more complicated but the energy transport equation exhibits the basic features of the three-dimensional case. In both instances, we obtain an integro-differential equation in which the integrals are evaluated along certain resonant curves. Since the resonant curves of the present investigation are straight lines, those wave vectors $\mathbf{k}_1$ collinear with $\mathbf{k}$, the final equation takes a surprisingly simple form.

This introduces a further striking contrast to the fully dispersive case. In the latter, the associated H theorem indicates that the energy is primarily redistributed over all $\mathbf{k}$ space by the resonance mechanism, since the resonant loci intersect. In the present case the resonance mechanism can only cause the energy to be shared along a given ray. Isotropic redistribution is achieved by a local transfer between neighbouring rays, and the transfer rate depends on the energy difference between adjacent members. This mechanism also takes longer to become operative.

Due to the different character of these waves, we call them semidispersive. In a sense, they take advantage of both their dispersive and non-dispersive character; the former promotes closure while the latter ensures the simplicity of the resonant loci.

We remark on an interesting consequence of the theory as applied to the shallow-water equations. These equations are themselves a model describing the behaviour of waves whose wavelength is large compared to the depth. When the energy has reached those wavelengths at which the shallow-water limit no longer applies, then further redistribution can only take place by the resonant mechanism associated with deep-water waves. However, this mechanism requires the mutual interaction of four waves and takes a much longer time to become operative. This would suggest a certain piling up of energy at certain scales.

Due to the strength of the secular terms which arise, it is reasonable to ask whether any energy exchange occurs when the physical situation is modelled by a description where the dependent variables, such as the surface elevation and velocity potential, themselves decay sufficiently rapidly at infinity to permit ordinary Fourier transforms. A naive perturbation expansion on the Fourier transform of the wave height produces non-uniformities which can be interpreted as an exchange of energy between various modes.

2. Analysis

The derivation of the hierarchy of cumulant equations has been carried out in many of the cited references (for example, Hasselmann 1962, Newell 1968) and for the present work we will merely write down the equations. As a first step, however, we introduce the following notation and definitions. Let

$$R^{(2)}(\mathbf{r}_1, t), R^{(3)}(\mathbf{r}_1, \mathbf{r}_2, t), \ldots, R^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N-1}, t)$$

represent the cumulants (which are in (one-to-one) correspondence with the moments) which correspond to correlations between values of the stochastic variable estimated at the spatial positions $\mathbf{x}, \mathbf{x}+\mathbf{r}_1, \mathbf{x}+\mathbf{r}_2, \ldots, \mathbf{x}+\mathbf{r}_{N-1}$. We stipulate

spatial homogeneity which means that the moments are functions only of the relative geometry and time t. We also remark that it is consistent with the equation hierarchy to take $R^{(1)}(t)$ to be zero for all time. It proves convenient to work with the Fourier transforms of the cumulants which are defined by the inverse transformation of

$$R^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N-1}, t)$$
$$= \int_{-\infty}^{\infty} Q^{(N)}(\mathbf{k}_1, \ldots, \mathbf{k}_{N-1}, t) \exp\left[i\mathbf{k}_1 . \mathbf{r}_1 + \ldots + i\mathbf{k}_{N-1} . \mathbf{r}_{N-1}\right] d\mathbf{k}_1 \ldots d\mathbf{k}_{N-1}, \quad (2.1)$$

where $Q^{(N)}$ are ordinary functions of the wave-numbers at the initial time. To take advantage of the weakness of the non-linear coupling, we remove the linear response of the system by introducing:

$$Q^{(N)}(\mathbf{k}_1, \ldots, \mathbf{k}_{N-1}, t) = \sum_{s_1 \ldots s_N} q^{(N)}(\mathbf{k}_1, s_1; \mathbf{k}_2, s_2; \ldots; \mathbf{k}_N, s_N)$$
$$\times \exp\left[i(s_1\omega_1 + s_2\omega_2 + \ldots + s_N\omega_N)\, t\right], \quad (2.2)$$

where $\mathbf{k}_1 + \ldots + \mathbf{k}_N = 0$ and the linear frequency $\omega(\mathbf{k}_r)$, an even function of $\mathbf{k}_r$ is denoted by ω_r. The parameters $s_1, \ldots, s_N$ are sign parameters and the summation is over all combinations of plus and minus. In the system we are discussing this corresponds to the fact that a given wave $\mathbf{k}$ can propagate in two directions. In systems where there are more than two frequencies corresponding to a given wave vector, the summation must be extended to include all these frequencies. Since in many cases the parameters $\mathbf{k}_j, s_j$ occur as a pair in the arguments of certain functions, we will denote this number pair by $\boldsymbol{\kappa}_j$. In places where this notation is ambiguous we will write the full argument dependence.

The first two numbers in the equations hierarchy for $q^{(N)}$ are

$$\frac{dq^{(2)}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', t)}{dt} = \epsilon P(0, 0') \sum_{s_l s_m} \int_{-\infty}^{\infty} L(\boldsymbol{\kappa}, \boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m)\, q^{(3)}(\boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m, \boldsymbol{\kappa}')$$
$$\times \exp\left[iW_{lm,0}t\right] \delta_{lm,0}\, d\mathbf{k}_{lm}, \quad \mathbf{k} + \mathbf{k}' = 0; \quad (2.3)$$

$$\frac{dq^{(3)}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}'', t)}{dt} = \epsilon P(0, 0', 0'') \sum_{s_l s_m} \int_{-\infty}^{\infty} L(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m) q^{(4)}(\boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m, \boldsymbol{\kappa}', \boldsymbol{\kappa}'', t)$$
$$\times \exp\left[iW_{lm,0}t\right] \delta_{lm,0}\, d\mathbf{k}_{lm} + 2\epsilon P(0, 0', 0'')$$
$$\times \sum_{s_l s_m} L(\mathbf{k}, s; -\mathbf{k}', s_l; -\mathbf{k}'', s_m)\, q^{(2)}(\mathbf{k}', s'; -\mathbf{k}', s_l) q^{(2)}(\mathbf{k}'', s''; -\mathbf{k}'', s_m)$$
$$\times \exp\left[i(s_l\omega(\mathbf{k}') + s_m\omega(\mathbf{k}'') - s\omega)\, t\right], \quad (2.4)$$

where $\mathbf{k} + \mathbf{k}' + \mathbf{k}'' = 0$ and $P(0, 0'), P(0, 0', 0'')$ denote the cyclic permutation over $(\mathbf{k}, \mathbf{k}'), (s, s')$; $(\mathbf{k}, \mathbf{k}', \mathbf{k}''), (s, s', s'')$ respectively. We have introduced the further notational contractions

$$\delta_{lm,0} = \delta(\mathbf{k}_l + \mathbf{k}_m - \mathbf{k}),$$
$$W_{lm,0} = s_l\omega_l + s_m\omega_m - s\omega, \quad \omega_l = \omega(\mathbf{k}_l),$$
$$d\mathbf{k}_{lm} = d\mathbf{k}_l d\mathbf{k}_m,$$

where $\delta(x)$ is the Dirac delta function. The coefficients $L(\boldsymbol{\kappa}, \boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m)$ depend on the particular physical situation; for sound waves (μ is the adiabatic constant)

$$L(\boldsymbol{\kappa}, \boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m) = \frac{ic_0^2}{4}\left\{\frac{\mathbf{k}.\mathbf{k}_l}{s_l\omega_l} + \frac{\mathbf{k}.\mathbf{k}_m}{s_m\omega_m} + \frac{s\omega}{s_l\omega_l s_m\omega_m}\mathbf{k}_l.\mathbf{k}_m\right\} + \frac{i}{4}(\mu - 2)s\omega. \tag{2.5}$$

It becomes clear that the nature of the exponent of the oscillation is crucial. We therefore divide the problem into three classes. The first class is that of non-dispersive waves for which the group velocity is a constant vector. In such cases the exponent of the oscillatory exponential can be exactly zero for all $\mathbf{k}$. For example, examine the integrand of (2.3) after setting $\omega = \boldsymbol{\alpha}.\mathbf{k}$, $\boldsymbol{\alpha}$ constant. Only for a few select s (sign values) will we get oscillatory cancellation. In the remaining terms the time scale can be changed to ϵt which returns the hierarchy to one of strongly interacting character. All hopes of a natural closure are thus lost.

The second class, the class of dispersive waves, have been dealt with in the literature; see, for example, Benney & Newell (1969). The distinguishing feature of previous analyses has been that the contribution to the long-time behaviour of the various quantities arose from those curves in $\mathbf{k}$ space where the exponent of the oscillatory terms has a simple zero. This class includes a wide variety of wave phenomena; Rossby waves, deep-water gravity waves, plasma waves (see Hasselmann 1962, Kenyon 1967, Davidson 1969, Rebhan & Wimmel 1969).

The third class arises when the exponent of the oscillatory term has a multiple zero. A principal reason for the study of such a class is that waves whose linear dispersion relation is

$$\omega \propto \pm|\mathbf{k}|$$

belong to this category. Included in this class are sound waves, shallow-water waves, electromagnetic waves in non-linear media, and certain types of plasma waves. For such cases the exponent $h(\mathbf{k}_1; \mathbf{k}) = s_1|\mathbf{k}_1| + s_2|\mathbf{k} - \mathbf{k}_1| - s|\mathbf{k}|$ is such that both $h(\mathbf{k}_1; \mathbf{k})$ and $\nabla_{\mathbf{k}_1} h(\mathbf{k}_1, \mathbf{k})$ can be zero simultaneously with a suitable choice of sign parameters. It will turn out that the cumulant hierarchy for this class of waves also has a natural closure. The reason for this is the cancellation produced by different wave groups crossing a given direction. Thus, whereas it is true that there is coherence along a given ray, the directional dependence of the group velocity results in sufficient incoherence to induce a natural asymptotic closure on the system.

Our approach is to apply the naive perturbation expansion

$$q^{(N)} = q_0^{(N)} + \epsilon q_1^{(N)} + \epsilon^2 q_2^{(N)} + \ldots \quad (N = 2, 3, \ldots), \tag{2.6}$$

to the system. Our goal will be to choose the slow variation of $q_0^{(N)}$ in order to remove non-uniformities from the expansion (2.6). We should note that the long-time limit ($t \to \infty$, $\epsilon^\alpha t$ fixed, some $\alpha > 0$) cannot always be taken in Fourier space and in these cases we must ask that the corresponding expansions in physical space remain uniformly ordered. The first terms in the energy

density $q^{(2)}(\mathbf{k}, s; -\mathbf{k}, -s)$ (which we may abbreviate to $q^{(2)}(\boldsymbol{\kappa})$) may be written

$$q_0^{(2)}(\boldsymbol{\kappa}) = q_0^{(2)}(\boldsymbol{\kappa}), \tag{2.7}$$

$$q_1^{(2)}(\boldsymbol{\kappa}) = P(0,-0) \sum_{s_l s_m} \int_{-\infty}^{\infty} L(\boldsymbol{\kappa}, \boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m)\, q_0^{(3)}(\boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m, -\boldsymbol{\kappa})\, \Delta(W_{lm,0})\, \delta_{lm,0}\, d\mathbf{k}_{lm}, \tag{2.8}$$

$$\begin{aligned} q_2^{(2)}(\boldsymbol{\kappa}) = {} & P(0,-0) \int_0^t dt \sum_{s_l s_m} \int_{-\infty}^{\infty} L(\boldsymbol{\kappa}, \boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m) \{P(-0, l, m) \\ & \times \sum_{s_p s_q} \int_{-\infty}^{\infty} L(-\boldsymbol{\kappa}, \boldsymbol{\kappa}_p, \boldsymbol{\kappa}_q)\, q_0^{(4)}(\boldsymbol{\kappa}_p, \boldsymbol{\kappa}_q, \boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m)\, \Delta(s_p, \omega_p + s_q \omega_p + s\omega)\, \delta_{pq,0}\, d\boldsymbol{\kappa}_{pq}\} \\ & \times \exp[iW_{lm,0}t]\, \delta_{lm,0}\, d\mathbf{k}_{lm} + 2P(0,-0) \int_0^t dt \sum_{s_l s_m} \int_{-\infty}^{\infty} L(\boldsymbol{\kappa}, \boldsymbol{\kappa}_l, \boldsymbol{\kappa}_m) \\ & \times \{2P(-0, l, m) \sum_{s_p s_q} L(-\mathbf{k}, -s; -\mathbf{k}_l, s_p; -\mathbf{k}_m, s_q)\, q_0^{(2)}(\mathbf{k}, s_l; -\mathbf{k}_l, s_p) \\ & \times q_0^{(2)}(\mathbf{k}_m, s_m; -\mathbf{k}_m, s_q)\, \Delta(s_p \omega_l + s_q \omega_m + s\omega)\} \exp[iW_{lm,0}t]\, \delta_{lm,0}\, d\mathbf{k}_{lm}, \end{aligned} \tag{2.9}$$

where $P(-0, l, m)$ represents the cyclic permutation $-\boldsymbol{\kappa} \to \boldsymbol{\kappa}_l \to \boldsymbol{\kappa}_m \to -\boldsymbol{\kappa}$, and $P(0,-0)$ is the permutation $\boldsymbol{\kappa} \to -\boldsymbol{\kappa}$.

The long-time behaviour of the integrals is characterized by the asymptotic behaviour of the integral

$$\int_{-\infty}^{\infty} F(\mathbf{k}_1; \mathbf{k})\, \Delta(h(\mathbf{k}_1; \mathbf{k}))\, d\mathbf{k}_1, \tag{2.10}$$

where $$h(\mathbf{k}_1; \mathbf{k}) = s_1|\mathbf{k}_1| + s_2|\mathbf{k} - \mathbf{k}_1| - s|\mathbf{k}|.$$

The major contribution to (2.10) arises when, in addition to

$$\nabla_{\mathbf{k}_1} h(\mathbf{k}_1; \mathbf{k}) = 0, \tag{2.11}$$

the sign parameters are chosen so that the function $h(\mathbf{k}_1; \mathbf{k})$ itself is identically zero on this locus. Thus the behaviour of (2.10) is governed to a large extent by the second variation of $h(\mathbf{k}_1; \mathbf{k})$ in the neighbourhood of the locus given by (2.11) and this depends on the dimension of the problem. We will first tackle the case of three dimensions as the re-ordering of the expansion (2.6) and the interpretation of the non-uniformities is easier there. In this case, the integral (2.10) reduces to an integral of the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, \frac{\exp[i(x^2+y^2)t] - 1}{i(x^2+y^2)}\, dx\, dy, \tag{2.12}$$

which, after using polar co-ordinates and setting $\rho = r^2 = x^2 + y^2$, is equivalent to

$$\int_0^{\infty} g(\rho)\, \frac{e^{i\rho t} - 1}{i\rho}\, d\rho. \tag{2.13}$$

The real part of (2.13)

$$\int_0^{\infty} g(\rho)\, \frac{\sin \rho t}{\rho}\, d\rho \to \tfrac{1}{2}\pi\, g(0) \operatorname{sgn} t \quad \text{as} \quad t \to \infty,$$

and thus we may say that in the long-time limit

$$\operatorname{Re}\Delta(\rho) \sim \tfrac{1}{2}\pi\,\delta(\rho)\operatorname{sgn} t, \tag{2.14}$$

where $\delta(\rho)$ is the Dirac delta function. This is reminiscent of the fully dispersive case where the long-time behaviour of the system is governed by integrals of the type

$$\int_{-\infty}^{\infty} g(\rho)\frac{e^{i\rho t}-1}{i\rho}\,d\rho. \tag{2.15}$$

However, since the integration path is $[0,\infty]$, the terms analogous to the Cauchy principal value are not present. In fact, we may write

$$\begin{aligned} -i\int_0^{\infty} g(\rho)\frac{\cos\rho t-1}{\rho}\,d\rho &= -i\int_0^{\infty}\frac{g(\rho)-g(0)\,e^{-a\rho}}{\rho}(\cos\rho t-1)\,d\rho \\ -ig(0)\int_0^{\infty} e^{-a\rho}\frac{\cos\rho t-1}{\rho}\,d\rho, \quad (a>0) &\xrightarrow{t\to+\infty} -ig(0)\log t+ig(0)\log a \\ &\quad +i\int_0^{\infty}\frac{g(\rho)-g(0)\,e^{-a\rho}}{\rho}\,d\rho \end{aligned} \tag{2.16}$$

using the Riemann–Lebesgue lemma and simple integration techniques. Thus the imaginary part of $\Delta(\rho)$ acts like a Dirac delta function which is weakly unbounded in time plus an order one term which takes account of the values of $g(\rho)$ over a finite range. That (2.12) indeed acts like $\log|t|$ for long time can also be verified by differentiating (2.12) with respect to t and evaluating the resultant integral in terms of Fresnel integrals. It can also be verified that for $t\to+\infty$,

$$-i\int_{-\infty}^{0} g(\rho)\frac{\cos\rho t-1}{\rho}\,d\rho \sim ig(0)\log t-ig(0)\log a+i\int_{-\infty}^{0}\frac{g(\rho)-g(0)\,e^{+a\rho}}{\rho}\,d\rho, \tag{2.17}$$

which, when added to (2.16), yields the familiar result

$$-i\int_{-\infty}^{\infty} g(\rho)\frac{\cos\rho t-1}{\rho}\,d\rho \sim i\mathrm{P}\int_{-\infty}^{\infty}\frac{g(\rho)}{\rho}\,d\rho \tag{2.18}$$

where P denotes the Cauchy principal value. In applying the limit $t\to\infty$ we find:

$$q_1^{(2)} \sim O(\log t, 1), \quad q_2^{(2)} \sim O(t, \log t, 1). \tag{2.19}$$

No $t\log t$ term appears in $q_2^{(2)}(\kappa)$ due to the reality of $q^{(2)}(\kappa)$ and the fact that we are dealing with a conservative system. We choose the slow temporal variation of $q_0^{(2)}(\kappa)$ in order to remove the strongest non-uniformities, namely the t growth in $q_{(2)}^{(2)}(\kappa)$. To be specific we define $T_2=\epsilon^2 t$ whence

$$\frac{d}{dt}\to\frac{\partial}{\partial t}+\epsilon^2\frac{\partial}{\partial T_2}$$

and we find

$$\begin{aligned} \frac{dq_0^{(2)}(\kappa)}{dT_2} = (\mu+1)^2\frac{c_0\pi^2}{4}\Bigg[&\int_k^{\infty} dk_{1x}k_{1x}(k_{1x}-k)\,G(k,k_{1x}) \\ &+\int^{k} dk_{1x}k_{1x}(k-k_{1x})\,G(k,k_{1x})+\int_{-\infty}^{0} dk_{1x}(-k_{1x})\,(k-k_{1x})\,H(k,k_{1x})\Bigg]. \end{aligned} \tag{2.20}$$

We have chosen the $\mathbf{k}_1$ axes so that the given $\mathbf{k}$ lies along the positive k_{1x} axis. When $\mathbf{k}$ lies along the negative k_{1x} axis we find the result by using the property

$$q^{(2)}(-\mathbf{k}, -s; \mathbf{k}, s) = q^{(2)}(\mathbf{k}, s; -\mathbf{k}, -s).$$

The reason that the right-hand side of (2.20) decomposes into three integrals is that for different values of the sign of $k_{1x}-k$ the choice of the sign parameters is different. In order to satisfy (2.11) and the condition that $h(\mathbf{k}_1; k)$ itself is zero we require that s_1, s_2 be chosen such that

$$s_1 \operatorname{sgn} k_{1x} = s_2 \operatorname{sgn}(k - k_{1x}) = s.$$

Thus for $k_{1x} > k > 0$ we choose $s_1 = -s_2 = s$; for $k > k_{1x} > 0$ we choose $s_1 = s_2 = s$; for $k > 0 > k_{1x}$ we choose $-s_1 = s_2 = s$. Since the coefficients $F(\mathbf{k}_1; \mathbf{k})$ in (2.10) generally depend on s_1, s_2 and s we must decompose (2.10) into three separate integrations. The functions $G(k_1 k_{1x})$ and $H(k_1 k_{1x})$ are

$$\begin{aligned} G(k, k_{1x}) = {} & k q^{(2)}(k_{1x}, s; -k_{1x}, -s)\, q^{(2)}(k - k_{1x}, s; k_{1x} - k, -s) \\ & - k_{1x} q^{(2)}(k, s; -k, -s)\, q^{(2)}(k - k_{1x}, s; k_{1x} - k, -s) \\ & - (k - k_{1x})\, q^{(2)}(k, s; -k, -s)\, q^{(2)}(k_{1x}, s; -k_{1x} - s) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} H(k, k_{1x}) = {} & k q^{(2)}(k_{1x}, -s; -k_{1x}, s)\, q^{(2)}(k - k_{1x}, s; k_{1x} - k, -s) \\ & - k_{1x} q^{(2)}(k, s; -k, -s)\, q^{(2)}(k - k_{1x}, s; k_{1x} - k, -s) \\ & - (k - k_{1x})\, q^{(2)}(k_1 s; -k, -s)\, q^{(2)}(k_{1x}, -s; -k_{1x}, s). \end{aligned} \tag{2.22}$$

We can write (2.20) in a more convenient form by setting $k_{1x} = \alpha k$ in the first integral, $k_{1x} = \gamma k$ in the second, $k_{1x} = -\gamma k$ in the third and also noting that

$$q^{(2)}(k_{1x}, s; -k_{1x}, -s) = q^{(2)}(-k_{1x}, -s; k_{1x}, s).$$

It is readily shown that the transformation

$$\beta - 1 = \gamma$$

on the first integral yields the third and so we obtain

$$\begin{aligned} \frac{dq(\mathbf{k}, s)}{dt} = {} & \epsilon^2(\mu+1)^2 \frac{c_0 \pi^2}{4} \left[2 \int_0^\infty k^4 \gamma(\gamma+1) \{ q(\gamma \mathbf{k}, s)\, q((\gamma+1)\mathbf{k}, s) \right. \\ & \left. + \gamma q(\mathbf{k}, s)\, q((\gamma+1)\mathbf{k}, s) - (\gamma+1) q(\mathbf{k}, s)\, q(\gamma \mathbf{k}, s) \} d\gamma \right] \\ & + \int_0^1 k^4 \alpha(1-\alpha) \{ q(\alpha \mathbf{k}, s)\, q((1-\alpha)\mathbf{k}, s) - \alpha q(\mathbf{k}, s)\, q((\mathbf{k}, s)\, 1-\alpha) \\ & - (1-\alpha) q(\mathbf{k}, s)\, q(\alpha \mathbf{k}, s) \} d\alpha, \end{aligned} \tag{2.23}$$

where for convenience we have written $q^{(2)}(\mathbf{k}, s; -\mathbf{k}, -s)$ as $q(\mathbf{k}, s)$. The energy density is proportional to

$$\int_{-\infty}^{\infty} (q^{(2)}(\mathbf{k}, +1) + q^{(2)}(\mathbf{k}, -1))\, d\mathbf{k}.$$

The result shows that the rate of change of $q(\mathbf{k}, +1)$ (or $q^{(2)}(\mathbf{k}, -1)$) can be obtained by a knowledge of $q(\mathbf{k}', +1)$ for all $\mathbf{k}'$ collinear with the given $\mathbf{k}$. Thus the equation for the energy density is closed.

3. The long-time behaviour of the higher cumulants

Before proceeding to the two-dimensional case, we wish to emphasize several crucial points. The asymptotic analysis used in obtaining (2.19) assumes that both $q_0^{(3)}$ and $q_0^{(4)}$ are smooth, or to be precise, absolutely integrable and continuous at least in the neighbourhood of the resonant surfaces. By assumption the cumulants are initially smooth; however, in order for the analysis to be consistent we must ensure that they remain smooth *to zeroth order*. This property depends on the nature of the strongest secular terms appearing in $q_i^{(3)}$ and $q_i^{(4)}$ $(i = 1, 2, 3, \ldots)$, which terms determine the slow temporal behaviour of $q_0^{(3)}, q_0^{(4)}$. In turn, to evaluate in a consistent manner the asymptotic behaviour of $q_i^{(3)}$, $q_i^{(4)}$ we must know the asymptotic behaviour of $q_i^{(N)}$. At any closure stage one needs some information on the behaviour of all the cumulants. This has been done by Benney & Newell (1969) for the case of full dispersive systems; in the present case the terms which contribute are formally the same but their asymptotic behaviour differs because of the double resonance. It turns out that the choice of the temporal behaviour of $q_0^{(N)}$ must be made as follows

$$\begin{aligned} \partial q_0^{(N)}/\partial T_2^1 &= q_0^{(N)} \sum_{i=1}^{N} F(\mathbf{k}_i, s_i), \\ \partial q_0^{(N)}/\partial T_2 &= q_0^{(N)} \sum_{i=1}^{N} G(\mathbf{k}_i, s_i), \quad \sum_{i=1}^{N} \mathbf{k}_i = 0, \end{aligned} \tag{3.1}$$

where $$q_0^{(N)} = q_0^{(N)}(\boldsymbol{\kappa}_1, \ldots, \boldsymbol{\kappa}_N), \quad \mathbf{k}_1 + \ldots + \mathbf{k}_N = 0,$$

$$T_2^1 = \epsilon^2 \int \log t \, dt, \quad T_2 = \epsilon^2 t;$$

or equivalently

$$\frac{d}{dt}(\log q_0^{(N)}) = \epsilon^2 \log t \sum_{i=1}^{N} F(\mathbf{k}_i, s_i) + \epsilon^2 \sum_{i=1}^{N} G(\mathbf{k}_i, s_i), \tag{3.2}$$

where

$$F(\mathbf{k}, s) = \frac{-i\pi c_0(\mu+1)^2}{4} k^4 \left[\int_1^\infty (\beta-1)^2 \beta q(\beta\mathbf{k}, s)\, d\beta - \int_0^1 (1-\alpha)^2 \alpha q(\alpha\mathbf{k}, s)\, d\alpha \right.$$
$$\left. - \int_0^\infty \gamma(\gamma+1)^2 q(\gamma\mathbf{k}, s) \right].$$

$G(\mathbf{k}, s)$ has a somewhat similar, but algebraically more complicated, form in which the integrals include non-resonant terms.

We have pointed out that the zeroth-order cumulants remain smooth. This is not the case for the $O(\epsilon)$ components. If we evaluate $Q_1^{(3)}$ we find that in order to take the long-time limit we must evaluate the integrals from the point of view of physical space. In this context, $Q_1^{(3)}$ has developed a generalized function behaviour. In fact, we can write the long-time behaviour of the triple cumulant in physical space as

$$R^{(3)}(\mathbf{r}, \mathbf{r}', t) \sim 2\epsilon \int_{-\infty}^{\infty} P(0, 0', 0'') \{L(\boldsymbol{\kappa}, -\boldsymbol{\kappa}', -\boldsymbol{\kappa}'') q_0^{(2)}(\boldsymbol{\kappa}') q_0^{(2)}(\boldsymbol{\kappa}'')\}$$
$$\times \exp[i\mathbf{k}.\mathbf{r} + i\mathbf{k}'.\mathbf{r}'] \bar{\Delta}(s\omega + s'\omega' + s''\omega'') \delta(\mathbf{k} + \mathbf{k}' + \mathbf{k}'') \, d\mathbf{k} \, d\mathbf{k}' \, d\mathbf{k}'', \tag{3.3}$$

where $\tilde{\Delta}$ is the sum of generalized functions one of which is weakly unbounded ($\sim \log t$). The first non-vanishing contributions after long time to $R^{(3)}$, $R^{(4)}$ and in general $R^{(N)}$ are $O(\epsilon)$, $O(\epsilon^2)$ and $O(\epsilon^{N-2})$ respectively. In physical space, the dominant time behaviour of the various cumulants can be written

$$R^{(2)} \sim O(1) + O(\epsilon \log t, \epsilon) + O(\epsilon^2 t, \epsilon^2) + \ldots, \tag{3.4}$$

$$R^{(3)} \sim O(1/t) + O(\epsilon \log t, \epsilon) + O(\epsilon^2 \log t) + O(\epsilon^3 t \log t, \epsilon^3 t, \ldots), \tag{3.5}$$

$$R^{(4)} \sim O(1/t^2) + O(\epsilon/t) + O(\epsilon^2 \log t) + \ldots. \tag{3.6}$$

Removing the $\epsilon^2 t$ non-uniformity in (3.4) we obtain equation (2.23); this choice also removes the $\epsilon^3 t \log t$ and $\epsilon^3 t$ non-uniformities which appear in $R_3^{(3)}$. The remaining asymptotic expansions are uniformly valid for times $\epsilon^2 t = O(1)$ or $T_2 \geqslant O(1)$. Equation (2.23) has an associated H theorem which ensures that

$$P(0, 0', 0'')\{L(\boldsymbol{\kappa}, -\boldsymbol{\kappa}', -\boldsymbol{\kappa}'') q_0^{(2)}(\boldsymbol{\kappa}', \epsilon^2 t) q_0^{(2)}(\boldsymbol{\kappa}'', \epsilon^2 t)\} \to 0, \quad \epsilon^2 t \to \infty$$

and thus removes the non-uniformity appearing as $\log t$ in $R^{(3)}$. Thus to $O(\epsilon^2)$ the system achieves joint Gaussian characteristics. The remaining non-uniformities of the $\log t$ variety are only non-uniformities on time scales transcendentally long compared with those of direct interest. Nevertheless, their existence suggests that $q^{(2)}$ be expanded as $q_0^{(2)} + \epsilon \log \epsilon^2 \hat{q}_1^{(2)} + \epsilon q_1^{(2)} + \ldots$,

$$\begin{aligned}\hat{q}_1^{(2)}(\mathbf{k}, s) = \mathrm{Re}\Big\{\frac{s\pi(\mu+1)}{2}\Big[&\int_k^\infty k_{1x}(k_{1x}-k)\, q_0^{(3)}(k_{1x}\hat{k}, s;\ \mathbf{k}-\mathbf{k}_{1x}\hat{k}, -s; -\mathbf{k}, -s)\, dk_{1x} \\ &+ \int_0^k k_{1x}(k-k_{1x})\, q_0^{(3)}(k_{1x}\hat{k}, s;\ \mathbf{k}-k_{1x}, \mathbf{k}, s;\ -\mathbf{k}_1 - s)\, dk_{1x} \\ &+ \int_{-\infty}^0 k_{1x}(k_{1x}-k)\, q_0^{(3)}(k_{1x}\hat{k}, -s;\ \mathbf{k}-k_{1x}\hat{k}, s;\ -\mathbf{k}, -s)\, dk_{1x}\Big]\Big\},\end{aligned}$$

where $\hat{k}$ is the unit vector in the direction $\mathbf{k}$ and k is $|\mathbf{k}|$. Then

$$\epsilon \log \epsilon^2 \hat{q}_1^{(2)} + \epsilon q_1^{(2)} \to O(\epsilon \log T_2, \epsilon) \quad \text{as} \quad T_2 = O(1).$$

Long before such terms become important, another, physically more important, class of non-uniformities arise. Mathematically they occur as the first subdominant terms in the asymptotic expansions of steepest-descent-type integrals; they contain derivatives of the energy densities in planes perpendicular to the resonant rays. Physically, they are responsible for a further smoothing of spectral energy in $\mathbf{k}$ space; the resonant interactions only serve to equidistribute the energy along individual rays. These effects become important on the $1/\epsilon^4$ time scale.

As a final comment we emphasize that the above results have been derived from a time $t = 0$ when the cumulants were smooth. Extreme care must be taken if the initial-value problem is to be done from some other time $t_1 = O(1/\epsilon^2)$. In this situation, the evolution of the $O(\epsilon)$ generalized function structure in $Q^{(3)}$ (see expansion for $R^{3)}$) must be taken into account. A similar problem, which gave rise to an irretraceability paradox, has been discussed and resolved in the case of fully dispersive systems and we refer the reader to the work of Benney & Newell

(1969). Briefly stated the situation is as follows: the initial problem from $t = 0$ has secular terms arising from integrals containing the argument

$$\int_0^t \frac{e^{ihs}-1}{ih}\,ds.$$

However, beginning from t_1, the initial values for $q_0^{(3)}$ contain an $O(\epsilon)$ cusp-like structure of the form $(e^{iht_1}-1)/ih$ and non-uniformities in $q_2^{(2)}(\kappa)$ are produced by the term

$$\frac{e^{iht_1}-1}{ih}\int_{t_1}^t e^{ih(s-t_1)}\,ds$$

in addition to the term

$$\int_{t_1}^t \frac{e^{ih(s-t_1)}-1}{ih}\,ds.$$

Addition of these two terms yields

$$\int_0^t \frac{e^{ihs}-1}{ih}\,ds - \int_0^{t_1} \frac{e^{ihs}-1}{ih}\,ds.$$

The choice of the long-time-scale behaviour of $\partial q_0^2(\kappa, T_2)/\partial T_2$ is then consistent with that choice made when beginning at $t = 0$.

4. The two-dimensional case

The two-dimensional case yields an energy transport equation essentially of the same type as (2.23). However, the choice of time scales becomes more subtle and the long-time behaviour of the singular integrals is significantly different. The behaviour of the system is again characterized by integrals of the form (2.10) except that now we consider the wave vectors as two dimensional. In this case, the integral (2.10) reduces to an integral of the form

$$\int_{-\infty}^{\infty} f(x)\,\frac{e^{i\beta x^2 t}-1}{i\beta x^2}\,dx. \tag{4.1}$$

A straightforward integration by parts yields that the dominant behaviour of the above integral for long times is

$$2t\int_{-\infty}^{\infty} f(x)\,e^{i\beta x^2 t}dx$$

which tends to

$$2t(\pi/i\beta t)^{\frac{1}{2}}f(0)+O(1), \quad \text{as} \quad t\to\infty. \tag{4.2}$$

Thus the real and imaginary parts of the Δ function act as Dirac delta functions. Due to the lower dimension the response of $\Delta(h)$ is stronger by a factor of $t^{\frac{1}{2}}$. This means the re-ordering of the non-uniformities becomes rather subtle as apparent secularities arising in lower-order (in ϵ) terms are ignored in favour of

more dominant ones arising in highest-order (in ϵ) terms. To be specific, we will schematically write down the dominant behaviour of the various cumulants.

$$R^{(2)} \sim O(1) + O(\epsilon t^{\frac{1}{2}}, \epsilon) + O(\epsilon^2 t^{\frac{3}{2}}, \epsilon^2 t^{\frac{1}{2}}, \epsilon^2) + \ldots, \tag{4.3}$$

$$R^{(3)} \sim O(1/t^{\frac{1}{2}}) + O(\epsilon t^{\frac{1}{2}}) + O(\epsilon^2 t) + O(\epsilon^2 t^2) + \ldots, \tag{4.4}$$

$$R^{(4)} \sim O(1/t) + O(\epsilon) + \ldots, \tag{4.5}$$

$$R^{(N)} \sim (1/t^{\frac{1}{2}(N-2)}) + \ldots. \tag{4.6}$$

At first sight it would appear that it is necessary to remove the secular term $\epsilon t^{\frac{1}{2}}$ from $R^{(2)}$, whose terms involve integrals whose integrands contain the zeroth-order approximation to the third-order cumulants denoted $q_0^{(3)}$. However, from the point of view of maintaining a uniform expansion in time, it is the $\epsilon^2 t^{\frac{3}{2}}$ term which is the first to become secular. Thus even though the term appears at higher order in ϵ its effect is felt to order one in a time $t \sim O(1/\epsilon^{\frac{4}{3}})$ which is short compared with $t \sim O(1/\epsilon^2)$, the time scale on which the $\epsilon t^{\frac{1}{2}}$ secularity first becomes non-uniform. This is fortunate as regards the closure of the system since the $\epsilon^2 t^{\frac{3}{2}}$ term contains only products of energy densities.

Nevertheless, it might yet be argued that in a sense the triple cumulant $R^{(3)}$ becomes disordered on a time scale $t = O(1/\epsilon)$. This is only true, however, if we expect that the behaviour of this cumulant is dominated by the first term. But we know this is not the case in either dispersive wave systems or in three-dimensional semidispersive waves where an order one (in the former, in the latter order $\log t$) contribution was allowed to exist in the ϵ term, whereas the zeroth-order terms decayed at least as fast as $1/t$. The crucial point to emphasize, therefore, is that the higher cumulants have two classes of terms; ones which vanish on the short-time scale and others which remain finite until later times. The $t^{\frac{1}{2}}$ term in $R^{(3)}$ must only be considered secular compared to unity, namely for those times when $\epsilon t^{\frac{1}{2}} \geqslant 1$. Moreover, the coefficient involved in this term is the integrand which appears in the $\epsilon^2 t^{\frac{3}{2}}$ terms in $R^{(2)}$. Therefore the coefficient (a product of energy densities) will be a function of the time scale $\epsilon^2 t^{\frac{3}{2}}$ and its behaviour (determined by the solution to the energy transport equation) will play a decisive role in the boundedness or non-boundedness of $R^{(3)}$ on the $t \sim O(1/\epsilon^2)$ time scale. It turns out that due to the solution of the integro-differential energy transport equation, the spectrum relaxes to a state where the terms multiplying the $\epsilon t^{\frac{1}{2}}$ term vanish and so $R^{(3)}$ is indeed bounded for all time. If this were not the case, the re-ordering process at the later time would have to undergo slight modifications. These would be necessary in any event, as we shall see in the following paragraph.

The relevant multiple-time-scale sequence to achieve a first closure is therefore

$$t = t, \quad T_2 = \tfrac{2}{3}\epsilon^2 t^{\frac{3}{2}}.$$

With this choice, we must remark that a function of T_2 is only a slowly varying function of time for times $\epsilon^2 t^{\frac{1}{2}} \ll 1$ or for $T_2 \ll 1/\epsilon^4$ which is much longer than the time scale of the first closure $T_2 = O(1)$, $(t = O(1/\epsilon^{\frac{4}{3}}))$. Note that a secular growth of ϵe^t could not be taken care of in this fashion. We also emphasize that at the time scale $T_2 = O(1)$, the errors in the first- or zeroth-order approximations to

$R^{(2)}$ are $O(\epsilon^{\frac{1}{3}})$. This suggests that the asymptotic sequence initially used to order the cumulants $(1, \epsilon, \epsilon^2, \ldots, \text{etc.})$ is no longer relevant and a different sequence is required. This leads naturally to the notion of a matched asymptotic expansions approach and one could well use these ideas by viewing the whole problem from the point of view of physical space since, in this context, all the fast-time behaviour (oscillations) has been removed. Nonetheless, it is more convenient to use the multiple scale approach as long as possible

For brevity, we will omit the details of the asymptotic expansions and simply quote the results obtained by choosing dq_0/dT_2 in order to suppress the $\epsilon^2 t^{\frac{3}{2}}$ non-uniformity. We obtain

$$\begin{aligned}\frac{dq_0(\mathbf{k}, s;\ T_2)}{dT_2} = {} & \tfrac{9}{4} c_0^{\frac{3}{2}} \pi^{\frac{1}{2}} k^{\frac{7}{2}} \Big[2 \int_0^\infty d\gamma [\gamma(1+\gamma)]^{\frac{1}{2}} \\ & \times \{ q_0(\gamma \mathbf{k}, s)\, q_0((\gamma+1)\,\mathbf{k}, s) + \gamma q_0(\mathbf{k}, s)\, q_0((\gamma+1)\,\mathbf{k}, s) \\ & - (\gamma+1) q_0(\mathbf{k}, s)\, q_0(\gamma \mathbf{k}, s) \} + \int_0^1 d\alpha [\alpha(1-\alpha)]^{\frac{1}{2}} \{ q_0(\alpha \mathbf{k}, s)\, q_0((1-\alpha)\mathbf{k}, s) \\ & - \alpha q_0(\mathbf{k}, s)\, q_0((1-\alpha)\,\mathbf{k}, s) - (1-\alpha)\, q_0(\mathbf{k}, s)\, q_0(\alpha \mathbf{k}, s) \} \Big]. \end{aligned} \tag{4.7}$$

Again we remind the reader that the choice (4.7) must be consistent with a uniform ordering of all the moments and in particular we must again ensure the continued smoothness of the zeroth-order approximation of the higher cumulants. Similar remarks about the nature of the initial-value problem from time $T_2 = O(1)$ apply as in the three-dimensional case.

We close this section with a brief note about the case when the dependent variables permit ordinary Fourier transforms. The fact that we have double resonances means that a perturbation series on the Fourier transform of the dependent variable itself (say $u(\mathbf{x}, t)$) will become non-uniform in time and therefore there is an energy transfer process associated with the continuous case. We will just treat the two-dimensional case and set

$$a^s(k, t) = a_0^s(k, T) + \epsilon a_1^s(k, t, T) + \ldots, \quad T = 2\epsilon t^{\frac{1}{2}}, \tag{4.8}$$

where

$$u(\mathbf{x}, t) = \int A(\mathbf{k}, t) \exp[i\mathbf{k}.\mathbf{x}]\, d\mathbf{k} = \int \sum_s a^s(\mathbf{k}, t) \exp[i\mathbf{k}.\mathbf{x} + is\omega t]\, d\mathbf{k}.$$

We find that due to non-uniformities in a_1^s we must choose

$$\begin{aligned}\frac{da_0^s(k, T)}{dT} = {} & 3i(\tfrac{1}{2}\pi c_0)^{\frac{1}{2}} k^{\frac{5}{2}} \Big[s e^{\frac{1}{4}is\pi} \int_0^1 a_0^s(\alpha \mathbf{k})\, a_0^s((1-\alpha)\,\mathbf{k})\, \alpha(1-\alpha)\, d\alpha \\ & + 2s e^{-\frac{1}{4}is\pi} \int_0^\infty a_0^s(\gamma \mathbf{k})\, a_0^s((\gamma+1)\,\mathbf{k})\, \gamma(\gamma+1)\, d\gamma \Big]. \end{aligned} \tag{4.9}$$

The transfer equation (2.38) satisfies certain conservation properties, those of energy and momentum. We can show, in a manner similar to the analysis of §5, that

$$\sum_s \int_{-\infty}^{\infty} a_0^s(\mathbf{k})\, \overset{*}{a_0^s}(\mathbf{k})\, d\mathbf{k}$$

is conserved by the transport equation.

5. Conservation of energy, momentum and an H theorem

The energy transport equation derived in §§ 2 and 4 can conveniently be written

$$\begin{aligned} k^{p-1}\frac{dq(\mathbf{k})}{d\tau} = 2\int_0^\infty & d\gamma(\gamma(\gamma+1)\,k^3)^{\frac{1}{2}(p-1)}\,k^3\{q(\gamma\mathbf{k})\,q((\gamma+1)\,\mathbf{k}) \\ & +\gamma q(\mathbf{k})\,q((\gamma+1)\,\mathbf{k})-(\gamma+1)\,q(\mathbf{k})\,q(\gamma\mathbf{k})\} \\ & +\int_0^1 d\alpha(\alpha(1-\alpha)\,k^3)^{\frac{1}{2}(p-1)}\,k^3\{q(\alpha\mathbf{k})\,q((1-\alpha)\,\mathbf{k}) \\ & -\alpha q(\mathbf{k})\,q((1-\alpha)\,\mathbf{k})-(1-\alpha)\,q(\mathbf{k})\,q(\alpha\mathbf{k})\}, \end{aligned} \tag{5.1}$$

where p is the dimension of the system, $q(\mathbf{k})$ is written for $q_0(\mathbf{k}, s)$ and τ is the modified slow time scale. An equilibrium solution of (5.1), which also removes the regenerated third-order cumulant (that part which did not decay to zero on the fast time scale), is

$$q(\mathbf{k}) = \text{constant}. \tag{5.2}$$

This choice relaxes the system closer to a state of exact (in ϵ) joint Gaussianity. The mode of energy redistribution is one of resonance but the interesting feature is that, to this time scale, no angular (in $\mathbf{k}$ space) transfer occurs; for example, the constant in (5.2) is in general a function of angle. Angular redistribution is achieved by a local transfer process which occurs on a longer time scale ($1/\epsilon^4$ in three dimensions; $1/\epsilon^{\frac{8}{3}}$ in two) by terms which are similar to the second closure terms derived by Benney & Newell (1969).

Conservation theorems for the transport equation (5.1) can readily be found by considering

$$\frac{dH_F}{d\tau} = \int_0^\infty F(\mathbf{k})\frac{dq(\mathbf{k})}{d\tau}\,k^{p-1}\,dk, \tag{5.3}$$

where k denotes $|\mathbf{k}|$. Using (5.1) we obtain,

$$\begin{aligned} \frac{dH_F}{d\tau} = 2\int_0^\infty\int_0^\infty & F(\mathbf{k})\{\gamma(1+\gamma)\,k^3\}^{\frac{1}{2}(p-1)}\,k^3\{q(\gamma\mathbf{k})\,q((\gamma+1)\,\mathbf{k}) \\ & +\gamma q(\mathbf{k})\,q((\gamma+1)\,\mathbf{k})-(1+\gamma)\,q(\mathbf{k})\,q(\gamma\mathbf{k})\}\,d\gamma\,dk \\ & +\int_0^\infty\int_0^1 F(\mathbf{k})\{(1-\alpha)\,k^3\}^{\frac{1}{2}(p-1)}\,k^3\{q(\alpha k)\,q((1-\alpha)\,\mathbf{k}) \\ & -aq(\mathbf{k})\,q((1-\alpha)\,\mathbf{k})-(1-\alpha)\,q(\mathbf{k})\,q(a\mathbf{k})\}\,d\alpha\,dk. \end{aligned} \tag{5.4}$$

In the second integral we make the transformation

$$\gamma = \alpha/(1-\alpha), \quad \mathbf{k} = (\gamma+1)\mathbf{l} \tag{5.5}$$

and obtain

$$\begin{aligned} \frac{dH_F}{d\tau} = \int_0^\infty\int_0^\infty & \{2F(\mathbf{k})-(\gamma+1)\,F((\gamma+1)\,\mathbf{k})\}\{q(\gamma\mathbf{k})\,q((\gamma+1)\,\mathbf{k}) \\ & +\gamma q(\mathbf{k})\,q((\gamma+1)\,\mathbf{k})-(\gamma+1)\,q(\mathbf{k})\,q(\gamma\mathbf{k})\}\,(\gamma(1+\gamma)\,k^3)^{\frac{1}{2}(p-1)}\,k^3\,dk. \end{aligned}$$

Writing the right-hand side as the sum of two equal terms and applying the transformation

$$\gamma = 1/\beta, \quad \mathbf{k} = \beta \mathbf{l} \tag{5.6}$$

to the second, we obtain on renaming dummy integration variables

$$\frac{dH_F}{d\tau} = \int_0^\infty \int_0^\infty \{F(\mathbf{k}) + \gamma F(\gamma \mathbf{k}) - (\gamma+1) F((\gamma+1)\mathbf{k})\} \left\{ \frac{1}{q(\mathbf{k})} + \frac{\gamma}{q(\gamma \mathbf{k})} - \frac{\gamma+1}{q((\gamma+1)\mathbf{k})} \right\} \times (\gamma(1+\gamma)\, k^3)^{\frac{1}{2}(p-1)}\, k^3 q(k)\, q(\gamma \mathbf{k})\, q((\gamma+1)\mathbf{k})\, d\gamma\, dk. \tag{5.7}$$

The choice $F = 1$ yields the conservation of energy and momentum. The choice $F = (q(k))^{-1}$ yields the result

$$\frac{d}{d\tau} \int_0^\infty \log\left(\frac{q(\mathbf{k})}{\nu}\right) \frac{1}{p}\, d(k^p) \geqslant 0, \tag{5.8}$$

where ν is some arbitrary renormalization. Thus, $\log [q(\mathbf{k})/\nu]$ is an increasing function. Unfortunately the integral diverges for most realizations of the spectrum. Nevertheless, (5.8) is strongly suggestive; at the initial time we expect that since the second correlation decays rapidly with distance most of the contribution to H_F comes from the zero value of $q(k)$ at large k in which case $H_F = -\infty$. The subsequent response of the system to a non-equilibrium state is toward a smoothing of the power spectrum and in particular a movement of energy to high k. This can be seen by simply examining the integrand of (5.1) and is also consistent with the H theorem. This was also verified by numerical experiment (Aucoin 1970). The process can first relax when the energy (and momentum) is equally divided between the cells in $\mathbf{k}$ space. This is equivalent to the Rayleigh–Jeans spectrum.

Before verifying this latter statement, we remark that the $\int \log [q(k)/\nu]\, d\mathbf{k}$ spectrum has a universal character for wave systems; it is also the entropy functional for fully dispersive waves. It is of interest to show, at least formally, that the equilibrium solutions of such systems are relative maxima. Given the sequence of independent system constraints,

$$\int f_i(\mathbf{k})\, q(\mathbf{k})\, d\mathbf{k} = \alpha_i \quad (i = 1, 2, \ldots, n), \tag{5.9}$$

we ask for the form of $q(\mathbf{k})$ for which the entropy functional is maximal subject to the constraints (5.9). We obtain that

$$q(\mathbf{k}) = 1 \Big/ \sum_{i=1}^{n} \lambda_i f_i(\mathbf{k}), \tag{5.10}$$

where the λ_i are Lagrange multipliers whose values may be determined from the α_i. As an example, we quote the case of a system of Rossby waves (which are fully dispersive and which have as independent constraints the constancy of energy $\int q(\mathbf{k})\, dk$ and mean squared vorticity $\int k^2 q(\mathbf{k})\, dk$). The H theorem then suggests that $1/\lambda_1 + \lambda_2 k^2$ is the form of the energy spectrum. In the present case there is

only one constraint (energy and momentum are the same along a ray) and thus we expect the system to tend towards the equipartition solution

$$q(k) = 1/\lambda_1. \tag{5.11}$$

For a system with finite energy we might expect that in a finite time the energy redistributes itself evenly over a range of wave vectors; the behaviour at high k is probably described by a decaying tail. For instance, the solution $q \propto 1/\tau k^{\frac{1}{2}(p+5)}$ formally satisfies (5.1). Unfortunately, the integrals diverge; however, we note two facts which may have significance. The first is that the divergence is only logarithmic (there is some cancellation of singularities); the second is that the transfer for this spectrum is completely local.

In accordance with the wave particles analogy we may set

$$q(\mathbf{k}) = \nu\omega(\mathbf{k})\, n(\mathbf{k}), \tag{5.12}$$

where $n(\mathbf{k})$ is the number of wave groups between $\mathbf{k}$ and $\mathbf{k}+d\mathbf{k}$ and $\nu\omega(\mathbf{k})$ is the energy of each group. We find from (5.11) that

$$n(\mathbf{k}) \propto 1/\omega(\mathbf{k}) \tag{5.13}$$

which is the Rayleigh–Jeans spectrum. We remark, in addition, that wave groups are analogous to particles obeying Bose–Einstein rather than Fermi–Dirac statistics. The former corresponds to the case when the number of particles greatly exceeds the number of available states.

An irreversibility has entered the system. The generic equations usually have the property that if $v(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$ are solutions, then so is $-\mathbf{v}(x, -t)$, $\rho(\mathbf{x}, -t)$. (In this instance we are using the dependent variables of velocity and density corresponding to sound waves.) This is equivalent to saying that if $q_0^{(2)}(\boldsymbol{\kappa}, t)$ is a solution then so is $q_0^{(2)}(\boldsymbol{\kappa}, -t)$. This property is conserved about the initial time $t = 0$ where the cumulants are smooth; from (2.14) we note that if we go backward in time we must choose the negative sign. Once we have decided which side of the $t = 0$ starting point we are, the sign of (2.23) and (5.1) and consequently the direction of the H theorem is specified. If we wish to begin at some other point in time, say $t_1 = +O(1/\epsilon^2)$, then the initial-value problem has extra terms and again we obtain the positive sign. The energy transport equation can be used going forward or backward in time for $t > 0$ as long as we do not recross the time boundary layer at $t = 0$, although it is to be expected that due to the nature of the solution (the smoothing out of the power spectrum) the retracing of solutions becomes increasingly difficult.

The irreversibility has been introduced by a limit process ($t \to \infty$, $\epsilon^\gamma t$ fixed, $\gamma = 2, \frac{4}{3}$) and not by a probabilistic assumption. As has been noted, even an initially Gaussian state introduces a generalized function structure on the higher cumulants. Nonetheless, we emphasize that the dynamically evolved non-smooth structure is the only non-smoothness we can handle and for which (5.1) is valid. The system is unstable to finite amplitude perturbations (order ϵ) of a cusp-like nature. Moreover, as the system relaxes closer to its equilibrium point, one might expect that the magnitude of generalized function perturbations (in Fourier space) required to interrupt the progress of the solution towards

equilibrium becomes smaller. However, such perturbations are regarded as improbable (but not impossible) occurrences as they require an exact reversal of all the ensemble members. In this sense, the H theorem is, for a given experiment, a statement regarding the most likely progress of the system.

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